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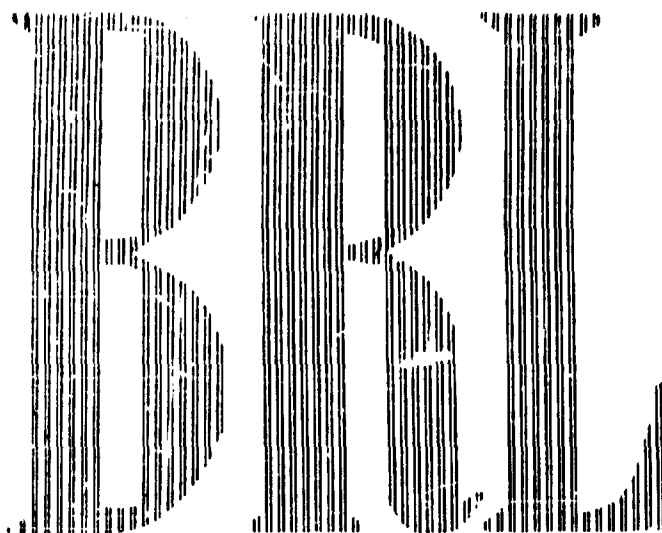
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MEMORANDUM REPORT NO. 1509
SEPTEMBER 1963

A DYNAMIC PROGRAMMING SOLUTION OF A
MISSILE ALLOCATION PROBLEM

William Sacco

RDT & E Project No. 1M010501A003 and 1M023201A098
BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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William Sacco

Computing Laboratory

RDT & E Project No. 1M010501A003 and 1M025201A098

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BALLISTIC RESEARCH LABORATORIES

MEMORANDUM REPORT NO. 1509

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Aberdeen Proving Ground, Md.
September 1963

A DYNAMIC PROGRAMMING SOLUTION OF A
MISSILE ALLOCATION PROBLEM

ABSTRACT

A generalization of a missile allocation problem proposed by Piccariello¹ is formulated using dynamic programming techniques. The formulation leads to an efficient algorithm for computing integer solutions to the discrete problem.

INTRODUCTION

H. Piccariello¹ has considered the following interesting missile allocation problem. Consider a target complex consisting of N missile launch sites and M control centers. It is assumed that if one (or more) control center survives an attack, then all surviving missiles can be launched and if all control centers are destroyed then none of the surviving missiles can be launched. Furthermore the assumption is made that each of the $N + M$ missile sites and control centers presents a separate target to an enemy missile attack. This is equivalent to the assumption that an enemy missile can destroy no target other than the one against which it is directed. The problem is to find an allocation of T identical missiles that when attacking the target complex will minimize the expected number of missiles capable of being launched.

Piccariello gives a solution to the problem when it is considered in its continuous form (allocations are not restricted to integer values). He also investigates the discrete form of the problem and shows by example, that, in general, the solution in the continuous case is not a solution for the discrete case. He then derives a necessary condition for the existence of a solution for the discrete case to differ from the continuous case.

In this paper a generalization of Piccariello's problem is formulated using dynamic programming techniques. The formulation leads to an efficient algorithm for computing integer solutions to the discrete problem.

THE FORMULATION OF THE PROBLEM

We shall use the following notations and definitions:

- P_i : probability of survival of missile site i ; $i = 1, 2, \dots, N$,
when attacked by a single incoming missile,
- \bar{P}_j : probability of survival of control center j ; $j = 1, 2, \dots, M$
when attacked by a single incoming missile,
- a_i : number of missiles at missile site i ,
- T : total number of attacking missiles;
- $E(X, Y)$: expected number of surviving missiles after an attack by
 T missiles, where $Y = T - X$ missiles are allocated to the
control centers and X missiles are allocated to the
missile sites.

The following expression for $E(X,Y)$ can be given:

$$E(X,Y) = \sum_{i=1}^N a_i P_i^{x_i} \left[1 - \prod_{j=1}^M (1 - \bar{P}_j^{y_j}) \right], \quad (1)$$

where x_i is the number of missiles allocated to the i th launcher site, y_j is the number of missiles allocated to the j th control center and

$$\sum_{i=1}^N x_i = X \quad (a)$$

$$\sum_{j=1}^M y_j = Y = T - X. \quad (b)$$

The objective is to find the minimum values of $E(X,Y)$ subject to the constraints a) and b) where each x_i and each y_j is an integer.

The above problem reduces to the Piccariello problem when $a_1 = a_2 = \dots = a_N = 1$, $P_1 = P_2 = \dots = P_N$, and $\bar{P}_1 = \bar{P}_2 = \dots = \bar{P}_M$.

DYNAMIC PROGRAMMING FORMULATION

An examination of equation (1), subject to the constraints (a) and (b) indicates that we are faced with an $N + M$ dimensional problem. In order to circumvent this dimensional difficulty, we will reformulate the problem as a multi-stage problem and apply the techniques of dynamic programming to obtain a feasible computational scheme. Let us define,

$s_k(X)$ = the minimum expected number of surviving missiles
for an allocation of X attacking missiles to k missile
sites,

and $f_l(Y)$ = the minimum value of the probability of the survival of at
least one control center for an allocation of Y attacking missiles
to l control centers.

Then we have

$$\text{Min } E(X, Y) = \text{Min}_{\{X, Y\}} g_N(X) f_M(Y) \quad (2)$$

subject to the constraint $X + Y = T$, where

$$g_N(X) = \text{Min}_{\{x_i\}} \sum_{i=1}^N a_i p_i^{x_i}; \quad x_1 + x_2 + \dots + x_N = X, \quad (3)$$

and

$$f_M(Y) = \text{Min}_{\{y_j\}} 1 - \prod_{j=1}^M (1 - \bar{p}_j^{y_j}) \quad \text{subject to the condition} \quad (4)$$

$$y_1 + y_2 + \dots + y_M = Y.$$

Using the Principle of Optimality² we can express the g_N recursively as

$$g_k(X) = \text{Min}_{x_k} \left[a_k p_k^{x_k} + g_{k-1}(X - x_k) \right], \quad k = 2, 3, \dots, N, \quad \text{where } x_k \text{ is permitted} \quad (5)$$

to vary over the set $\{0, 1, 2, \dots, X\}$. For $k - 1$ we have

$$g_1(X) = a_1 p_1^X \quad (6)$$

To obtain $f_M(Y)$ we first observe that $f_M(Y) = 1 - h_M(Y)$, where $h_M(Y) = \text{Max}_{\{y_j\}} \prod_{j=1}^M (1 - \bar{p}_j^{y_j})$.

Employing the Optimality Principle once again, we obtain

$$h_\ell(Y) = \text{Max}_{y_\ell} \left[(1 - \bar{p}_\ell^{y_\ell}) h_{\ell-1}(Y - y_\ell) \right], \quad \ell = 2, 3, \dots, M \quad \text{where } y_\ell \in \{0, 1, 2, \dots, Y\} \text{ and} \quad (7)$$

$$h_1(Y) = (1 - \bar{P}_1^Y). \quad (8)$$

The value Y is also permitted to range over the set of integers $\{0, 1, 2, \dots, T\}$.

COMPUTATIONAL PROCEDURE

The dynamic programming formulation imbeds the original problem within a family of analogous problems in which the basic parameters N , M , and T assume sets of values which permit us to obtain, in the course of the computation, the solution to a variety of sub-problems. Because of the structure of the process we are able to use the information obtained from sub-problems of the original problem to obtain the solution of the original problem. The input information that is required is the knowledge of N , M , and T and the values of the P_i 's, \bar{P}_j 's, and a_i 's.

We begin the computation by obtaining the sequences $\{g_k(X)\}$ and $\{x_k(X)\}$ from equations (5) and (6), and the sequences $\{h_p(Y)\}$ and $\{y_p(Y)\}$ from equations (7) and (8). Given this information, we are then prepared to use equation (2) to compute $\text{Min } E(X, Y)$.

NUMERICAL EXAMPLE

Let $N = 3$, $M = 2$, $T = 5$, $P_1 = .50$, $P_2 = .30$, $P_3 = .60$, $a_1 = 10$, $a_2 = 8$, $a_3 = 12$, $\bar{P}_1 = .60$, $\bar{P}_2 = .50$. From equations 5, 6, 7, and 8, we obtain the relations:

$$\begin{aligned} g_1(X) &= a_1 P_1^X, \\ g_2(X) &= \min_{x_2} \left\{ a_2 P_2^{x_2} + g_1(X - x_2) \right\}; \\ g_3(X) &= \min_{x_3} \left\{ a_3 P_3^{x_3} + g_2(X - x_3) \right\}; \end{aligned}$$

and

$$\begin{aligned} h_1(Y) &= (1 - \bar{P}_1^Y), \\ h_2(Y) &= \max_{y_2} \left\{ (1 - \bar{P}_2^{y_2}) h_1(Y - y_2) \right\}. \end{aligned}$$

Using the previous relations we obtain the values,

$$g_1(0) = 10$$

$$g_1(1) = 5$$

$$g_1(2) = 2.5$$

$$g_1(3) = 1.25$$

$$g_1(4) = .625$$

$$g_1(5) = .3125$$

$$g_2(0) = a_2 + g_1(0) = 8 + 10 = 18$$

$$g_2(1) = \text{Min} \left\{ \begin{array}{l} 8(.30)^0 + g_1(1) = 8 + 5 = 13 \\ 8(.30)^1 + g_1(0) = 2.4 + 10 = 12.4 \end{array} \right\} = 12.4$$

$$g_2(2) = \text{Min} \left\{ \begin{array}{l} 8 + g_1(2) = 10.5 \\ 2.4 + g_1(1) = 7.4 \\ .72 + g_1(0) = 10.72 \end{array} \right\} = 7.4$$

$$g_2(3) = \text{Min} \left\{ \begin{array}{l} 8 + g_1(3) = 9.25 \\ 2.4 + g_1(2) = 4.9 \\ .72 + g_1(1) = 5.72 \\ .216 + g_1(0) = 10.216 \end{array} \right\} = 4.9$$

$$g_2(4) = \text{Min} \left\{ \begin{array}{l} 8 + g_1(4) = 8.625 \\ 2.4 + g_1(3) = 3.65 \\ .72 + g_1(2) = 3.52 \\ .216 + g_1(1) = 5.216 \\ .0648 + g_1(0) = 10.0648 \end{array} \right\} = 3.52$$

$$g_2(5) = \text{Min} \left\{ \begin{array}{l} 8 + g_1(5) = 8.3125 \\ 2.4 + g_1(4) = 3.025 \\ .72 + g_1(3) = 1.97 \\ .216 + g_1(2) = 2.716 \\ .0648 + g_1(1) = 5.0648 \\ .01944 + g_1(0) = 10.01944 \end{array} \right\} = 1.97$$

$$g_3(0) = [12 + 10 + 8] = 30$$

$$g_3(1) = \text{Min} \left\{ \begin{array}{l} 12 + g_2(1) = 24.4 \\ 7.2 + g_2(0) = 25.2 \end{array} \right\} = 24.4$$

$$g_3(2) = \text{Min} \left\{ \begin{array}{l} 12 + 7.4 = 19.4 \\ 7.2 + 12.4 = 19.6 \\ 4.32 + 18 = 22.32 \end{array} \right\} = 19.4$$

$$g_3(3) = \text{Min} \left\{ \begin{array}{l} 12 + 4.9 = 16.9 \\ 7.2 + 7.4 = 14.6 \\ 4.32 + 12.4 = 16.72 \\ 2.592 + 18 = 20.592 \end{array} \right\} = 14.6$$

$$g_3(4) = \text{Min} \left\{ \begin{array}{l} 12 + 3.52 = 15.52 \\ 7.2 + 4.9 = 12.1 \\ 4.32 + 7.4 = 11.72 \\ 2.592 + 12.4 = 14.992 \\ 1.5552 + 18 = 19.5552 \end{array} \right\} = 11.72$$

$$g_3(5) = \text{Min} \left\{ \begin{array}{l} 12 + 1.97 = 13.97 \\ 7.2 + 3.52 = 10.72 \\ 4.32 + 4.9 = 9.22 \\ 2.592 + 7.4 = 9.992 \\ 1.5552 + 12.4 = 13.9552 \\ .93312 + 18 = 18.93312 \end{array} \right\} = 9.22$$

$$h_1(0) = 0$$

$$h_1(1) = .40$$

$$h_1(2) = .64$$

$$h_1(3) = .784$$

$$h_1(4) = .8804$$

$$h_1(5) = .92224$$

$$h_2(0) = 0$$

$$h_2(1) = \text{Max} \left\{ \begin{array}{l} 0 \\ 0 \end{array} \right\} = 0$$

$$h_2(2) = \text{Max} \left\{ \begin{array}{l} 0 \\ (.50)(.40) = .20 \\ 0 \end{array} \right\} = .20$$

$$h_2(3) = \text{Max} \left\{ \begin{array}{l} 0 \\ (.50)(.64) = .32 \\ (.75)(.40) = .30 \\ 0 \end{array} \right\} = .32$$

$$h_2(4) = \text{Max} \left\{ \begin{array}{l} 0 \\ (.50)(.784) = .392 \\ (.75)(.64) = .48 \\ (.875)(.40) = .35 \\ 0 \end{array} \right\} = .48$$

$$h_2(5) = \max \left\{ \begin{array}{l} 0 \\ (.50)(.8804) = .4402 \\ (.75)(.784) = .588 \\ (.875)(.64) = .56 \\ (.9375)(.40) = .375 \\ 0 \end{array} \right\} = .588$$

We are finally ready to make use of eq. (2)

$$\min_{\{X,Y\}} E(Y,X) = \min_{\{X,Y\}} g_N(X) f_M(Y)$$

where

$$f_M(Y) = 1 - h_M(Y),$$

and

$$X + Y = T.$$

For our example $N = 3$, $M = 2$ so that

$$\min_{\{X,Y\}} E(Y,X) = \min_{\{X,Y\}} g_3(X) f_2(Y) \text{ i.e.,}$$

$$\min_{\{X,Y\}} E(Y,X) = \min_{\{X,Y\}} \left\{ \begin{array}{l} g_3(0) f_2(5) = (30)(.412) = 12.36 \\ g_3(1) f_2(4) = (24.4)(.52) = 12.688 \\ g_3(2) f_2(3) = (19.4)(.68) = 13.192 \\ g_3(3) f_2(2) = (14.6)(.80) = 11.68 \\ g_3(4) f_2(1) = (11.72)(1) = 11.72 \\ g_3(5) f_2(0) = (9.22)(1) = 9.22 \end{array} \right\} = 9.22$$

The minimum value is 9.22. The allocation of missiles which yields the minimum value is $X = 5$, $Y = 0$. What remains is the determination of the

optimal values of the x_i ; $i = 1, 2, 3$:

$$x_3 = x_3(X) = x_3(5) = 2,$$

$$x_2 = x_2(X - x_3) = x_2(3) = 1$$

$$x_1 = x_1(X - x_3 - x_2) = x_1(2) = 2$$

Therefore the optimal allocation policy is given by

$$(x_1, x_2, x_3, y_1, y_2) = (2, 1, 2, 0, 0).$$

DISCUSSION

The author is grateful to Mr. H. L. Merritt for many helpful discussions during the formulation and solution of this problem. Mr. Merritt has pointed out to the author several problem areas of interest in which this procedure would have direct application. For instance it would be possible by utilizing these methods to decide upon an optimum ratio of control center to missile sites. It would also be possible to use this methodology to determine whether it is desirable to harden ICRM sites, or to utilize the additional money planned to be spent on active or passive defense by building more undefended launching sites.

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